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Electromagnetic wave scattering by many conducting small particles

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Abstract

A rigorous theory of electromagnetic wave scattering by small perfectly conducting particles is developed. The limiting case when the number of particles tends to infinity is discussed.

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1. Introduction

Assume that there are many small bodies D_m, $1 \leqslant m \leqslant M$, distributed in a bounded domain $D \subset \mathbb{R}^3$, $ka \ll 1$, $a = \frac{1}{2}\max_m \operatorname{diam} D_m$, $D_m \subset D$, $d = \min_{m \neq j} \operatorname{dist}(D_m, D_j) \gg a$. The body D_m has dielectric permittivity ϵ_m, ϵ_0 is the dielectric permittivity of the space. The case $\epsilon_m = \infty$ corresponds to a perfectly conducting small body. If D_m is a conductor then $\epsilon'_m = \epsilon_m + \mathrm{i}\frac{\sigma_m}{\omega}$ is its complex permittivity, σ_m is the conductivity and ω is the frequency. The scattering problem consists of finding the solution to the equations

$$\nabla \times E = \mathrm{i}\omega\mu H, \qquad \nabla \times H = -\mathrm{i}\omega\epsilon' E, \tag{1.1}$$

$$H = H_0 + h, \qquad E = E_0 + e, \tag{1.2}$$

where H_0, E_0 is the incident field,

$$E_0 = \mathbf{e_x}\xi\, \mathrm{e}^{\mathrm{i}kz}, \qquad H_0 = \frac{\nabla \times E_0}{\mathrm{i}\omega\mu_0}, \qquad k = \omega\sqrt{\epsilon_0\mu_0}, \tag{1.3}$$

μ_0 is the permeability of the free space, $\xi > 0$ is the amplitude of the incident field E_0, $\xi = |E_0|$, $\mathbf{e_x}$, $\mathbf{e_y}$, $\mathbf{e_z}$ is the Cartesian orthonormal basis, e and h are scattered fields which satisfy the radiation condition,

$$\epsilon' = \epsilon'_m = \epsilon_m + \mathrm{i}\frac{\sigma_m}{\omega} \quad \text{in } D_m, \qquad \epsilon' = \epsilon_0 \quad \text{in } \mathbb{R}^3 \Big\backslash \bigcup_{n=1}^{M} D_m,$$

M is the number of small bodies. Across the boundary S_m of D_m the tangential component of E and the normal component of H are continuous. Note that since μ_0 is constant, the normal component of $B := \mu_0 H$ and that of H are continuous simultaneously. Our basic assumptions are

$$ka \ll 1, \qquad \frac{a}{d} \ll 1. \tag{1.4}$$

The results of this paper are of two types:

(a) First, we derive asymptotic formulae for the scattered field at the distances greater than d from small bodies (particles) when M is not very large say $M = \mathcal{O}(10)$; these formulae can be used for a numerical solution of many-body scattering problems;
(b) Secondly, we derive an approximate equation for the effective field in the medium consisting of many small particles, $M = M(a) \to \infty$ as $a \to 0$.

Our derivations are valid for particles of arbitrary shape, not necessarily randomly distributed. For spherical particles with radius a formulae are simpler. The results are based on the author's theory of wave scattering by small particles of arbitrary shapes [5, 6].

In section 2 the many-body scattering problem for electromagnetic (EM) wave scattering is studied in the case $M = \mathcal{O}(10)$. In section 3 this problem is studied when $M = M(a) \to \infty$ as $a \to 0$ under suitable assumptions on the growth of $M(a)$, $M(a) = \mathcal{O}(a^{-3})$ as $a \to 0$.

We derive an integral equation for the effective field in the medium consisting of many small particles.

There is a large literature on scattering by small bodies. The theory was originated by Rayleigh in 1871, who understood that the dipole radiation is the basic part of the scattered field if the scatter is small, i.e., $ka \ll 1$ [1, 2]. However, only in [3–6], analytical formulae were derived for the S matrix for acoustic and electromagnetic wave scattering by small bodies of arbitrary shapes. These formulae allow one to calculate polarizability tensors with any desired accuracy analytically for bodies of arbitrary shapes. Assumptions (1.4) allow one to consider a small body D_m as if it was placed in an electrostatic constant field $E_e(x_m)$, where $x_m \in D_m$ is an interior point of D_m and E_e is the electric field acting on D_m and created by the incident field $E_0(x_m)$ together with the field scattered by all other bodies.

The many-body wave scattering theory for acoustic waves was developed in [7–9, 12–14]. This theory is basic for a method for creating materials with a desired refraction coefficient in acoustics, in particular, materials with a desired wave focusing property [10, 11], and a material with negative refraction, i.e., the material in which the group velocity is directed opposite to the phase velocity.

In [15] the EM wave scattering by many small inhomogeneities was studied under the assumption that these inhomogeneities were smooth perturbations of the dielectric parameter $\epsilon'(x)$, vanishing outside of a union of many small non-intersecting balls of radius a, centered at the points $x_m \in D$. The main results in [15] were:

(a) An asymptotically, as $a \to 0$, rigorous reduction of the many-body EM wave scattering problem to solving a linear algebraic system of equations, bypassing the usage of integral equations;
(b) A derivation of an equation for the effective (self-consistent) electric field E_e in the limiting medium, which was obtained by embedding a large number $M(a)$ of small particles in the region D, $M(a) = \mathcal{O}(a^{-(3-\kappa)})$, $a \to 0$, $0 < \kappa < 3$ is a constant.

The novel points of the present paper are:

The small particles are assumed well conducting, the corresponding perturbation of ϵ' is not necessarily smooth, and the boundary conditions on the boundaries of the small particles are taken into account by using general formulae for the polarization dipole moment, induced on the small particle by the exterior static field E_e. These formulae were derived in [4–6].

There are many papers and books dealing with wave propagation in random media (see [16] and references therein). These results are not used in this paper and their assumptions do not hold: we do not assume our particles necessarily randomly distributed.

2. EM wave scattering, $M = \mathcal{O}(10)$

Let us look for the solution of problems (1.1), (1.2) of the form

$$E(x) = E_0(x) + \sum_{m=1}^{M} \nabla \times \int_{S_m} g(x,t)\mathcal{J}^{(m)}(t)\,\mathrm{d}t, \qquad g(x,y) := \frac{\mathrm{e}^{\mathrm{i}k|x-y|}}{4\pi|x-y|}, \tag{2.1}$$

where the unknown vectors $\mathcal{J}^{(m)}$, $1 \leqslant m \leqslant M$, should be chosen so that the boundary conditions on S_m, $1 \leqslant m \leqslant M$, are satisfied. Outside of the union of the small bodies Maxwell's equations (1.1) are satisfied by function (2.1) for any $\mathcal{J}^{(m)}$ and $H = \frac{\nabla\times E}{\mathrm{i}\omega\mu_0}$. It follows from (2.1) that in the region $|x - x_m| \geqslant d \gg a$, $1 \leqslant m \leqslant M$, the field $E(x)$ can be well approximated by the formula:

$$E(x) = E_0(x) - \mathrm{i}\omega \sum_{m=1}^{M} \nabla \times (g(x, x_m)P^{(m)}), \qquad |x - x_m| \geqslant d \gg a, \tag{2.2}$$

where we have used the formula:

$$\int_{S_m} \mathcal{J}^{(m)}(t)\,\mathrm{d}t = -\mathrm{i}\omega P^{(m)}, \tag{2.3}$$

where $P^{(m)}$ is the induced on D_m dipole moment.

Indeed, if $\rho(x)$ is the density of the electric charge, then

$$P^{(m)} := \int_{D_m} x\rho_m(x)\,\mathrm{d}x.$$

From the conservation of the charge one gets $-\mathrm{i}\omega\rho_m + \nabla\cdot\mathcal{J}^{(m)} = 0$. Therefore

$$-\mathrm{i}\omega P^{(m)} = -\int_{D_m} x\nabla\cdot\mathcal{J}^{(m)}\,\mathrm{d}x = \int_{D_m} \mathcal{J}^{(m)}\,\mathrm{d}x - \int_{S_m} sN_j\mathcal{J}_j^{(m)}\,\mathrm{d}s, \tag{2.4}$$

where over the repeated indices j summation is understood and N_j is the jth component of the unit normal N to S_m directed out of D_m. We assume that $N\cdot\mathcal{J}^{(m)} = 0$ on S_m. If the depth of the skin layer is negligible compared with the size of D_m, then one may assume that the current $\mathcal{J}^{(m)}$ is concentrated on the surface $S^{(m)}$, so $\int_{D_m}\mathcal{J}^{(m)}\,\mathrm{d}x = \int_{S_m}\mathcal{J}^{(m)}\,\mathrm{d}t$, so formula (2.3) is obtained.

If $k \ll 1$, then one may assume that the small body D_m is placed in a constant effective field

$$E_e(x_m) := E_0(x_m) - \mathrm{i}\omega \sum_{m'\neq m} \nabla \times (g(x, x'_m)P^{(m')})|_{x=x_m}. \tag{2.5}$$

The induced dipole moment $P^{(m)}$ is calculated by the formula

$$P_i^{(m)} = \alpha_{ij}^{(m)}\epsilon_0 V^{(m)} E_{ej}(x_m) \qquad 1 \leqslant i \leqslant 3, \qquad 1 \leqslant m \leqslant M, \tag{2.6}$$

where summation is understood over the repeated indices j, $V^{(m)}$ is the volume of D_m, $V^{(m)} = \mathcal{O}(a^3)$ and $\alpha_{ij}^{(m)}$ is the polarizability tensor of the body D_m. Analytical formulae for this tensor for homogeneous dielectric bodies of arbitrary shapes were derived in [5, 6], so that one may consider the polarizability tensors $\alpha_{ij}^{(m)} := \alpha^{(m)}$ of the bodies D_m as known for the bodies D_m of arbitrary shapes. Formulae (2.5) and (2.6) reduce the many-body scattering problem to finding the unknown vectors $E_e(x_m)$ from the linear algebraic system:

$$E_e(x_m) = E_0(x_m) - \mathrm{i}\omega\nabla \times \sum_{m' \neq m}^{M} g(x, x_{m'})|_{x=x_m} \alpha^{(m')} \epsilon_0 V^{(m')} E_e(x_{m'}) \qquad 1 \leqslant m \leqslant M. \quad (2.7)$$

If the bodies D_m are balls of radius a, then $\alpha^{(m)} = 3\frac{\epsilon'_m - \epsilon_0}{\epsilon'_m + 2\epsilon_0}\delta_{ij}$, where δ_{ij} is the Kronecker symbol, ϵ'_m is the (complex) dielectric permittivity of D_m and ϵ_0 is the dielectric permittivity of the space exterior to the union of the small particles.

Let us show that the error of formula (2.2) is negligible. The omitted error term in (2.2) is equal to

$$-\mathrm{i}\omega\nabla \times \int_{S_m} (g(x, t) - g(x, x_m))\mathcal{J}^{(m)}(t)\,\mathrm{d}t = o(a^3), \qquad a \to 0. \quad (2.8)$$

Indeed, using assumptions (1.4), the inequality $|t - x_m| \leqslant a$, valid for $t \in S_m$, the estimates $\int_{S_m} \mathcal{J}^{(m)}(t)\,\mathrm{d}t = \mathcal{O}(a^3)$ and $V^{(m)} = \mathcal{O}(a^3)$, and formulae (2.3) and (2.6), one obtains:

$$|g(x, t) - g(x, x_m)| \leqslant c \max\left(\frac{ka}{d}, \frac{a}{d^2}\right).$$

Thus, term (2.8) is negligible compared with the term $-\mathrm{i}\omega\nabla \times (g(x, x_m)P^{(m)})$ in the region $|x - x_m| \geqslant d \gg a$.

This concludes the discussion of many-body EM wave scattering when $M = \mathcal{O}(10)$.

3. EM wave scattering, $M \to \infty$

Suppose that the number of small particles is very large. This number M in a bounded region D is of the order $\mathcal{O}\left(\frac{1}{d^3}\right)$. Therefore, if $d \gg a$, then $M = \mathcal{O}\left(\frac{1}{d^3}\right) \ll \mathcal{O}\left(\frac{1}{a^3}\right)$. Thus, to satisfy the assumption $a \ll d$ in (1.4) one has to assume that the total volume of the small particles is negligible: this volume is of the order

$$\mathcal{O}(Ma^3) \ll \mathcal{O}\left(\frac{1}{a^3}a^3\right) = \mathcal{O}(1).$$

If $M \to \infty$ in such a way that $a \to 0$ and (1.4) holds, then we replace the sum in (2.7) by the integral and get for the limiting field the equation

$$E_e(x) = E_0(x) - \mathrm{i}\omega\epsilon_0\nabla \times \int_D g(x, y)\alpha(y)N(y)E_e(y)\,\mathrm{d}y, \quad (3.1)$$

where $\alpha(y)$ is the average value of the polarizability tensor at the point y,

$$\alpha(y) := \lim_{\mathrm{diam}\Delta(y) \to 0} \frac{1}{\mathcal{N}(\Delta(y))} \sum_{x_m \in \Delta(y)} \alpha^{(m)},$$

where $\Delta(y)$ is a cube centered at the point $y, \mathcal{N}(\Delta(y))$ is the number of small particles in $\Delta(y)$ and $N(y)$ defines the number $\mathcal{N}(\Delta)$ of small particles in any subdomain $\Delta \subset D$ by the formula

$$\mathcal{N}(\Delta) := \sum_{x_m \in \Delta} 1 = \frac{1}{\gamma a^3} \int_\Delta N(y)\,\mathrm{d}y[1 + \eta(a)], \qquad \eta(a) = o(1), \quad a \to 0. \quad (3.2)$$

Here γ = constant, γ is defined by the formula $V^{(m)} = \gamma a^3$. Thus, $Ma^3 \ll \mathcal{O}(1)$ if and only if $N(y) \ll 1$. Let us explain how formula (3.1) is derived. Its derivation is based on the following lemma.

Lemma 3.1. *Assume that $f(x)$ and $N(x)$ are continuous functions in a bounded domain D and the points x_m are distributed in D so that (3.2) holds. Then there exists the limit:*

$$\lim_{a\to 0}\sum_{m=1}^{M} f(x_m)\gamma a^3 = \int_D f(x)N(x)\,\mathrm{d}x. \tag{3.3}$$

Proof. Let us partition D into a union of cubes Δ_j, $D = \cup_j \Delta_j$, where $\Delta_j := \Delta_j(y^{(j)})$ are cubes with no interior points of intersection, $y^{(j)}$ is the center of cube Δ_j, $|\Delta_j|$ is the volume of Δ_j and $\max_j\ \mathrm{diam}\Delta_j := b$. One has

$$\begin{aligned}\sum_{m=1}^{M} f(x_m)\gamma a^3 &= \sum_j f(y^{(j)})(1+\eta_1(b))\gamma a^3 \sum_{x_m\in\Delta_j} 1\\ &= \sum_j f(y^{(j)})N(y^{(j)})|\Delta_j|(1+\eta_1(b))(1+\eta(a)),\end{aligned} \tag{3.4}$$

where $\lim_{b\to 0}\eta_1(b) = 0$, $\lim_{a\to 0}\eta(a) = 0$, and we have used formula (3.2). Passing first to the limit $a \to 0$, and then taking $b \to 0$, one gets (3.3) because

$$\sum_j f(y^{(j)})N(y^{(j)})|\Delta_j| \to \int_D f(y)N(y)\,\mathrm{d}y$$

as $b \to 0$. Indeed, the sum on the left is a Riemann sum for the integral on the right. Lemma 3.1 is proved.

$\square$

Applying this lemma to sum (2.7) and assuming $V^{(m)} = \gamma a^3$, one gets equation (3.1). The smaller are the parameters $\frac{a}{d}$ and ka, the more accurately equation (3.1) describes the limiting field in the medium. The smallness of the parameter $\frac{a}{d}$ implies the smallness of $N(x)$ as we have already mentioned above. It follows from (3.1) that in the region D the following equation holds:

$$\nabla\times\nabla\times E_e = k^2 E_e - \mathrm{i}\omega\epsilon_0\nabla\times(\alpha(x)N(x)E_e(x)).$$

Thus, the limiting field E_e satisfies a Maxwell's equation in the region D and in this region the term

$$-\mathrm{i}\omega\epsilon_0\nabla\times(\alpha(x)N(x)E_e(x)), \qquad |N(x)| \ll 1$$

can be interpreted as the current.

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